Proceedings of the Jangjeon Mathematical Society
25 (2022), No. 1, pp. 43 – 58

www.jangjeon.or.kr
http://dx.doi.org/10.17777/pjms2022.25.1.43

EXTENDED BETA, HYPERGEOMETRIC AND CONFLUENT HYPERGEOMETRIC FUNCTIONS VIA MULTI-INDEX MITTAG-LEFFLER FUNCTION

NABIULLAH KHAN, MOHD AMAN AND TALHA USMAN

Abstract. In this paper, we first introduce an interesting generalization of the well-known beta function by means of a multi-index Mittag-Leffler function and study its important properties. We establish several new representations of the extended beta function, which shows explicit connections with other special functions. With the help of this new beta function, we introduce extensions of generalized hypergeometric function and confluent hypergeometric functions. Consequently, we present a systematic study of the fundamental properties of an extended hypergeometric function introduced here.

1. Introduction and preliminaries

Special functions constitute an important tool of mathematical analysis. They have become essential for scientists in many branches of Applied Mathematics dealing with the application of differential equations and a variety of other problems as well. The development of these branches has led to the introduction of new classes of special functions and their extensions and generalizations.
We emphasize that almost all the special functions can be represented in terms of generalized hypergeometric function and consequently, several interesting extensions and generalizations for the beta, hypergeometric and other functions can be found in the literature (see [1]-[13],[14, 16]) due to their tremendous applications. Motivated by the aforementioned work, we first define a new extension of beta function and study its useful properties and representations. Further, we obtain beta distribution and some statistical formulas. We then, using an extended beta function $B_{p,q}^{\lambda_i,\mu_i;\sigma,\tau}(x,y)$, study the extended forms of hypergeometric and confluent hypergeometric functions.

Throughout the paper, the letters $\mathbb{C}$, $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{Z}_0^-$ denote the sets of complex numbers, real numbers, positive real numbers and non-positive integers respectively, and let $\mathbb{R}_0^+ := \mathbb{R} \cup \{0\}$. The definitions given below are crucial to derive results in the paper.

The classical gamma function $\Gamma(z)$ (see [1]) is defined by

$$\Gamma(z) := \int_0^\infty e^{-t} t^{z-1}\, dt \quad (\Re(z) > 0). \tag{1.1}$$

Submission Date: 14-11-2018
2000 *Mathematics Subject Classification.* 33B15, 33C05, 33C15, 33E12 33E20.
Key words and phrases. Gamma function, Beta function, Hypergeometric function, Confluent hypergeometric function, Mellin transformation, Beta Distribution.

The beta function (see [1]) for a pair of complex numbers x and y with positive real part through the integral is given by

$$\begin{aligned} B(x,y) &= \int_0^1 t^{x-1}\,(1-t)^{y-1}dt \quad (\Re(x)>0\,,\,\Re(y)>0),\\ &= \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \frac{(x-1)!\,(y-1)!}{(x+y-1)!} \quad (x,y\notin \mathbb{Z}_0^-). \end{aligned} \tag{1.2}$$

The classical Gauss's hypergeometric function (see [1]) is defined by

$${}_2F_1\left[\begin{matrix} a, & b; \\ c; & \end{matrix}\; z\right] = \sum_{n=0}^{\infty} \frac{(a)_n\,(b)_n}{(c)_n}\,\frac{z^n}{n!}, \tag{1.3}$$

where $(a)_n$ $(a\in\mathbb{C})$ is the well known Pochhammer symbol. The well-known generalized hypergeometric function ${}_pF_q$ $(p,q\in\mathbb{N}_0)$ defined by (see [1, 16])

$${}_pF_q\left[\begin{matrix} a_1,\ldots,a_p; \\ b_1,\ldots,b_q; \end{matrix}\; z\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_q)_n}\,\frac{z^n}{n!}.$$

The confluent hypergeometric function (see [1]) is given by the series representation

$${}_1F_1(a;b;\;z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n}\,\frac{z^n}{n!}. \tag{1.4}$$

Chaudhry *et al.* [2] defined the extended beta function as follows:

$$B_p(x,y) = \int_0^1 t^{x-1}\,(1-t)^{y-1}\,\exp\left(-\frac{p}{t(1-t)}\right)dt \quad (\Re(x)>0\,,\,\Re(y)>0), \tag{1.5}$$

where $\Re(p)>0$ and parameters x and y are arbitray complex numbers.

The extended hypergeometric and confluent hypergeometric functions (see [3]) are defined respectively by

$$F_p(x,y,\omega;z) = \sum_{n=0}^{\infty} \frac{B_p(y+n,\omega-y)}{B(y,\omega-y)}\,\frac{z^n}{n!}, \tag{1.6}$$

$$(p\geq 0,\ \Re(\omega)>\Re(y)>0 \text{ and } |z|<1)$$

and

$$\Phi_p(y;\omega;z) = \sum_{n=0}^{\infty} \frac{B_p(y+n,\omega-y)}{B(y,\omega-y)}\,(x)_n\,\frac{z^n}{n!}, \tag{1.7}$$

$$(p\geq 0 \text{ and } \Re(\omega)>\Re(y)>0).$$

In terms of Mittag-Leffler function, Shadab *et al.* [14] introduced another extended beta function

$$B_p^{\lambda}(x,y) = \int_0^1 t^{x-1}\,(1-t)^{y-1}\,E_\lambda\left(-\frac{p}{t(1-t)}\right)dt \quad (\Re(x)>0\,,\,\Re(y)>0), \tag{1.8}$$

where $E_\lambda(.)$ is the classical Mittag-Leffler function defined as (see [1])

$$E_\lambda(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(\lambda n + 1)}. \tag{1.9}$$

We are concerned with the following generalized Mittag-Leffler function:

Let $m > 1$ be an integer, $\lambda_1, \dots, \lambda_m > 0$ and $\mu_1, \dots, \mu_m$ be arbitrary real(complex) numbers. By means of these multi-indices, Kiryakova [10] explored the multi-index Mittag-Leffler function defined by

$$E_{\left(\frac{1}{\lambda_i}\right),(\mu_i)}(x) := E^{(m)}_{\left(\frac{1}{\lambda_i}\right),(\mu_i)}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(\mu_1 + \frac{n}{\lambda_1}) \cdots \Gamma(\mu_m + \frac{n}{\lambda_m})}. \tag{1.10}$$

We in this paper, for $m > 1$ and $\forall \mu_i = 1$, also consider a special case of the multi-index Mittag-Leffler function defined by

$$E_{\lambda_i}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(\lambda_1 n + 1) \cdots \Gamma(\lambda_m n + 1)}, \tag{1.11}$$

for $\lambda_1 \dots \lambda_m$ be arbitrary real(complex) numbers. From now on, it is obvious to consider the index i $(i = 1, \dots, m)$.

Lemma 1.1. *The following relation of Mittag-Leffler function is used to derive results in the paper (see* [16]*):*

$$\int_0^\infty t^{a-1} E^{\delta}_{\frac{1}{\lambda_i},\mu_i}(-wt)dt = \frac{\Gamma(a)\Gamma(\delta - a)}{\Gamma(\delta)\, w^a\, \Gamma(\mu_i - a/\lambda_i)}, \tag{1.12}$$

which, for $\delta = w = 1$, becomes

$$\int_0^\infty t^{a-1} E_{\frac{1}{\lambda_i},\mu_i}(-t)dt = \frac{\Gamma(a)\Gamma(1 - a)}{\Gamma(\mu_i - a/\lambda_i)}. \tag{1.13}$$

2. An extended beta function

Our proposed extension of the beta function is defined by

$$B^{\lambda_i,\mu_i;\sigma,\tau}_{p,q}(x, y) = \int_0^1 t^{x-1} (1 - t)^{y-1} E_{\left(\frac{1}{\lambda_i}\right),(\mu_i)}\left(-\frac{p}{t^\sigma}\right) E_{\left(\frac{1}{\lambda_i}\right),(\mu_i)}\left(-\frac{q}{(1 - t)^\tau}\right) dt, \tag{2.1}$$

where $\Re(x) > 0,\ \Re(y) > 0,\ \Re(p) \geq 0,\ \Re(q) \geq 0,\ \lambda_i, \mu_i, k > 0$ and $E_{\left(\frac{1}{\lambda_i}\right),(\mu_i)}$ is a two parameter multi-index Mittag-Leffler function defined in (1.10).

Remark 2.1. *For $\sigma = \tau = \lambda_i = \mu_i (i = 1) = 1$,* (2.1) *reduce to definition* [6]*. For all $\lambda_i, \mu_i = 1$, $p = q$ and $\sigma = \tau$,* (2.1) *reduce to the extended beta function* [11] *and for $\sigma = \tau = \lambda_i = 1$, $p = 0 = q$, it reduces to the classical beta function given by* (1.2).

Integral representation of $B_{p,q}^{\lambda_i,\mu_i;\sigma,\tau}(x,y)$

Theorem 1. *The following representations are true:*

$$B_{p,q}^{\lambda_i,\mu_i;\sigma,\tau}(x,y) = 2\int_0^{\frac{\pi}{2}} \cos^{2x-1}\theta\, Sin^{2y-1}\theta\, E_{\left(\frac{1}{\lambda_i}\right),(\mu_i)}\left(-\frac{p}{\cos^{2\sigma}\theta}\right) E_{\left(\frac{1}{\lambda_i}\right),(\mu_i)}\left(-\frac{q}{\sin^{2\tau}\theta}\right) d\theta, \tag{2.2}$$

$$B_{p,q}^{\lambda_i,\mu_i;\sigma,\tau}(x,y) = \int_0^{\infty} \frac{u^{x-1}}{(1+u)^{x+y}} E_{\left(\frac{1}{\lambda_i}\right),(\mu_i)}\left(-\frac{p(1+u)^{\sigma}}{u^{\sigma}}\right) E_{\left(\frac{1}{\lambda_i}\right),(\mu_i)}\left(-\frac{q}{(1+u)^{\tau}}\right) du, \tag{2.3}$$

$$B_{p,q}^{\lambda_i,\mu_i;\sigma,\tau}(x,y) = 2^{1-x-y} \times \int_{-1}^{1} (1+u)^{x-1}(1-u)^{y-1} E_{\left(\frac{1}{\lambda_i}\right),(\mu_i)}\left(-\frac{2^{\sigma}p}{(1+u)^{\sigma}}\right) E_{\left(\frac{1}{\lambda_i}\right),(\mu_i)}\left(-\frac{2^{\tau}q}{(1-u)^{\tau}}\right) du, \tag{2.4}$$

and

$$B_{p,q}^{\lambda_i,\mu_i;\sigma,\tau}(x,y) = (c-a)^{1-x-y} \times \int_{a}^{c} (u-a)^{x-1}(c-u)^{y-1} E_{\left(\frac{1}{\lambda_i}\right),(\mu_i)}\left(-\frac{p(c-a)^{\sigma}}{(u-a)^{\sigma}}\right) E_{\left(\frac{1}{\lambda_i}\right),(\mu_i)}\left(-\frac{q(c-a)^{\tau}}{(c-u)^{\tau}}\right) du. \tag{2.5}$$

$$(\Re(p) > 0,\ \Re(q) > 0;\ p \geq 0,\ q \geq 0;\ \lambda_i \geq 0;\ \Re(x) > 0,\ \Re(y) > 0).$$

Proof. Let $t = \cos^2\theta$, $t = \frac{u}{1+u}$, $t = \frac{1+u}{2}$, $t = \frac{u-a}{c-a}$ respectively in equations (2.1), we obtain the above representations. □

Remark 2.2. *On setting $\sigma = \tau = \lambda_i = 1$ and $p = q$, $\lambda_i = 1$, $\sigma = \tau$ respectively, the above results can yield the corresponding representaions in* [6] *and* [11]*. Further for $p = 0 = q$, $\sigma = \tau = \lambda_i = 1$, the results reduces to some well known results for the beta function $B(x,y)$.*

3. Properties of $B_{p,q}^{\lambda_i,\mu_i;\sigma,\tau}(x,y)$

Theorem 2. *The summation formula for $B_{p,q}^{\lambda_i,\mu_i;\sigma,\tau}(x,y)$ is given by*

$$B_{p,q}^{\lambda_i,\mu_i;\sigma,\tau}(x,1-y) = \sum_{n=0}^{\infty} \frac{(y)_n}{n!} B_{p,q}^{\lambda_i,\mu_i;\sigma,\tau}(x+n,1) \quad (\Re(p) > 0,\ \Re(q) > 0). \tag{3.1}$$

Proof. Using the generalized binomial theorem

$$(1-t)^{-y} = \sum_{n=0}^{\infty} (y)_n \frac{t^n}{n!} \quad (|t| < 1),$$

we can write (2.1) as

$$B_{p,q}^{\lambda_i,\mu_i;\sigma,\tau}(x,1-y)=\int_0^1\sum_{n=0}^{\infty}(y)_n\,\frac{t^{x+n-1}}{n!}\,E_{\left(\frac{1}{\lambda_i}\right),(\mu_i)}\left(-\frac{p}{t^{\sigma}}\right)\,E_{\left(\frac{1}{\lambda_i}\right),(\mu_i)}\left(-\frac{q}{(1-t)^{\tau}}\right)\,dt,$$

from which we can easily obtain (3.1) by interchanging the order of integration and summation. □

Theorem 3. *The extended beta function has the following infinite summation formula:*

$$B_{p,q}^{\lambda_i,\mu_i;\sigma,\tau}(x,y)=\sum_{n=0}^{\infty}B_{p,q}^{\lambda_i,\mu_i;\sigma,\tau}(x+n,y+1)\quad(\Re(p)>0,\Re(q)>0).\tag{3.2}$$

Proof. Using the relation

$$(1-t)^{y-1}=(1-t)^{y}\sum_{n=0}^{\infty}t^{n},$$

We obtain

$$B_{p,q}^{\lambda_i,\mu_i;\sigma,\tau}(x,y)=\int_0^1(1-t)^{y}\sum_{n=0}^{\infty}t^{x+n-1}E_{\left(\frac{1}{\lambda_i}\right),(\mu_i)}\left(-\frac{p}{t^{\sigma}}\right)\,E_{\left(\frac{1}{\lambda_i}\right),(\mu_i)}\left(-\frac{q}{(1-t)^{\tau}}\right)\,dt.$$

Interchanging the order of integration and summation in the last expression leads us to the desired result (3.2). □

Theorem 4. *We have the following functional relation for the extended beta function $B_{p,q}^{\lambda_i,\mu_i;\sigma,\tau}(x,y)$:*

$$B_{p,q}^{\lambda_i,\mu_i;\sigma,\tau}(x+1,y)+B_{p,q}^{\lambda_i,\mu_i;\sigma,\tau}(x,y+1)=B_{p,q}^{\lambda_i,\mu_i;\sigma,\tau}(x,y).\tag{3.3}$$

Proof. Solving L.H.S. of (3.3), we get

$$B_{p,q}^{\lambda_i,\mu_i;\sigma,\tau}(x+1,y)+B_{p,q}^{\lambda_i,\mu_i;\sigma,\tau}(x,y+1)$$
$$=\int_0^1\left\{t^{x}\,(1-t)^{y-1}+t^{x-1}\,(1-t)^{y}\right\}\,E_{\left(\frac{1}{\lambda_i}\right),(\mu_i)}\left(-\frac{p}{t^{\sigma}}\right)\,E_{\left(\frac{1}{\lambda_i}\right),(\mu_i)}\left(-\frac{q}{(1-t)^{\tau}}\right)\,dt,$$

and by further solving we get the desired reult. □

Theorem 5. *The following relation holds true:*

$$B_{p,q}^{\lambda_i,\mu_i;\sigma,\tau}(\eta,-\eta-n)=\sum_{k=0}^{n}\binom{n}{k}B_{p,q}^{\lambda_i,\mu_i;\sigma,\tau}(\eta+k,-\eta-k),\quad(n\in\mathbb{N}_0).\tag{3.4}$$

Proof. First, we write

$$B_{p,q}^{\lambda_i,\mu_i;\sigma,\tau}(x+1,y)+B_{p,q}^{\lambda_i,\mu_i;\sigma,\tau}(x,y+1)=B_{p,q}^{\lambda_i,\mu_i;\sigma,\tau}(x,y).$$

Now, on substituting $x=\eta$ and $y=-\eta-n$ above, we arrive at

$$B_{p,q}^{\lambda_i,\mu_i;\sigma,\tau}(\eta,-\eta-n)=B_{p,q}^{\lambda_i,\mu_i;\sigma,\tau}(\eta,-\eta-n+1)+B_{p,q}^{\lambda_i,\mu_i;\sigma,\tau}(\eta+1,-\eta-n).$$

Writing this formula recursively with $n=1,2,3,\ldots$, we obtain

$$B_{p,q}^{\lambda_i,\mu_i;\sigma,\tau}(\eta,-\eta-1)=B_{p,q}^{\lambda_i,\mu_i;\sigma,\tau}(\eta,-\eta)+B_{p,q}^{\lambda_i,\mu_i;\sigma,\tau}(\eta+1,-\eta-1),$$

$$B_{p,q}^{\lambda_i,\mu_i;\sigma,\tau}(\eta,-\eta-2)=B_{p,q}^{\lambda_i,\mu_i;\sigma,\tau}(\eta,-\eta)+2B_{p,q}^{\lambda_i,\mu_i;\sigma,\tau}(\eta+1,-\eta-1)+B_{p,q}^{\lambda_i,\mu_i;\sigma,\tau}(\eta+2,-\eta-2),$$

and so on. By continuing the process, we arrive at (3.4). □

Theorem 6. *The following Mellin transformation formula for the extended beta function $B_{p,q}^{\lambda_i,\mu_i;\sigma,\tau}(x,y)$ holds:*

$$M\left\{B_{p,q}^{\lambda_i,\mu_i;\sigma,\tau}(x,y);p\to r,q\to s\right\}=\frac{\pi^2}{sin(\pi r)\,sin(\pi s)\Gamma(\mu_i-r/\lambda_i)\Gamma(\mu_i-s/\lambda_i)}\,B(x+\sigma r,y+\tau s) \tag{3.5}$$

$$(\Re(r)>0,\ \Re(s)>0,\ \Re(x+r)>0,\ \Re(y+s)>0).$$

Proof. We begin by providing Euler's reflection formula

$$\Gamma(x)\,\Gamma(1-x)=\frac{\pi}{\sin(\pi x)}. \tag{3.6}$$

We start by applying the usual Mellin transform on (2.1) and get

$$M\left\{B_{p,q}^{\lambda_i,\mu_i;\sigma,\tau}(x,y);p\to r,q\to s\right\}$$
$$=\int_0^\infty\int_0^\infty p^{r-1}q^{s-1}\left\{\int_0^1 t^{x-1}(1-t)^{y-1}E_{\left(\frac{1}{\lambda_i}\right),(\mu_i)}\left(-\frac{p}{t^\sigma}\right)E_{\left(\frac{1}{\lambda_i}\right),(\mu_i)}\left(-\frac{q}{(1-t)^\tau}\right)dt\right\}dp\,dq.$$

Interchanging the order of integration, we have

$$M\left\{B_{p,q}^{\lambda_i,\mu_i;\sigma,\tau}(x,y);p\to r,q\to s\right\}$$
$$=\int_0^1 t^{x-1}(1-t)^{y-1}\left\{\int_0^\infty p^{r-1}E_{\left(\frac{1}{\lambda_i}\right),(\mu_i)}\left(-\frac{p}{t^\sigma}\right)dp.\int_0^\infty q^{s-1}E_{\left(\frac{1}{\lambda_i}\right),(\mu_i)}\left(-\frac{q}{(1-t)^\tau}\right)dq\right\}dt.$$

Now substituting $\frac{p}{t^\sigma}=u$ and $\frac{q}{(1-t)^\tau}=v$ above, we obtain

$$M\left\{B_{p,q}^{\lambda_i,\mu_i;\sigma,\tau}(x,y);p\to r,q\to s\right\}=\int_0^1 t^{x+\sigma r-1}(1-t)^{y+\tau s-1}$$
$$\times\left\{\int_0^\infty u^{r-1}E_{\left(\frac{1}{\lambda_i}\right),(\mu_i)}(-u)\,du.\int_0^\infty v^{s-1}E_{\left(\frac{1}{\lambda_i}\right),(\mu_i)}(-v)\,dv\right\}dt,$$

which, on using (1.13), we get

$$M\left\{B_{p,q}^{\lambda_i,\mu_i;\sigma,\tau}(x,y);p\to r,q\to s\right\}=\frac{\Gamma(r)\Gamma(1-r)}{\Gamma(\mu_i-r/\lambda_i)}\,\frac{\Gamma(s)\Gamma(1-s)}{\Gamma(\mu_i-s/\lambda_i)}\,B(x+\sigma r,y+\tau s),$$

Finally, on using (3.6) above, we get the required result (3.5). □

4. **A new beta distribution**

The beta distribution is very often used to model data. Here, we define the following new beta distribution in the form of (2.1) and obtain its mean, variance and moment generating function.

For $B_{p,q}^{\lambda_i,\mu_i;\sigma,\tau}(x,y)$, the beta distribution is given by

$$f(t) = \begin{cases} \frac{1}{B_{p,q}^{\lambda_i,\mu_i;\sigma,\tau}(x,y)} t^{x-1}(1-t)^{y-1} E_{\left(\frac{1}{\lambda_i}\right),(\mu_i)}\left(-\frac{p}{t^\sigma}\right) E_{\left(\frac{1}{\lambda_i}\right),(\mu_i)}\left(-\frac{q}{(1-t)^\tau}\right) & (0 < t < 1), \\ 0 & \text{otherwise.} \end{cases} \tag{4.1}$$

For any real number ν, we have

$$E(X^\nu) = \frac{B_{p,q}^{\lambda_i,\mu_i;\sigma,\tau}(x+\nu,y)}{B_{p,q}^{\lambda_i,\mu_i;\sigma,\tau}(x,y)} \tag{4.2}$$

$$(p > 0,\ q > 0,\ -\infty < x < \infty,\ \infty < y < y).$$

When $\nu = 1$, we get the mean as a particular case of (4.2)

$$\mu = E(X) = \frac{B_{p,q}^{\lambda_i,\mu_i;\sigma,\tau}(x+1,y)}{B_{p,q}^{\lambda_i,\mu_i;\sigma,\tau}(x,y)}, \tag{4.3}$$

and the variance of the distribution is defined by

$$\sigma^2 = E(x) - \{E(X)\}^2 = \frac{B_{p,q}^{\lambda_i,\mu_i;\sigma,\tau}(x,y)\, B_{p,q}^{\lambda_i,\mu_i;\sigma,\tau}(x+2,y) - \left\{B_{p,q}^{\lambda_i,\mu_i;\sigma,\tau}(x+1,y)\right\}^2}{\left\{B_{p,q}^{\lambda_i,\mu_i;\sigma,\tau}(x,y)\right\}^2}. \tag{4.4}$$

The moment generating function of the distribution is defined by

$$M(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} E(X^n) = \frac{1}{B_{p,q}^{\lambda_i,\mu_i;\sigma,\tau}(x,y)} \sum_{n=0}^{\infty} B_{p,q}^{\lambda_i,\mu_i;\sigma,\tau}(x+n,y)\frac{t^n}{n!}. \tag{4.5}$$

The cumulative distribution is given by

$$F(x) = \frac{B_{z,p,q}^{\lambda_i,\mu_i;\sigma,\tau}(x,y)}{B_{p,q}^{\lambda_i,\mu_i;\sigma,\tau}(x,y)}, \tag{4.6}$$

where

$$B_{z,p,q}^{\lambda_i,\mu_i;\sigma,\tau}(x,y) = \int_0^z t^{x-1}(1-t)^{y-1} E_{\left(\frac{1}{\lambda_i}\right),(\mu_i)}\left(-\frac{p}{t^\sigma}\right) E_{\left(\frac{1}{\lambda_i}\right),(\mu_i)}\left(-\frac{q}{(1-t)^\tau}\right) dt, \tag{4.7}$$

$$(p > 0,\ q > 0,\ \lambda_i, \mu_i\sigma, \tau > 0,\ -\infty < x, y < \infty)$$

is a new extension of incomplete beta function.

5. **Hybrid Representations of** $B_{p,q}^{\lambda_i,\mu_i;\sigma,\tau}(x,y)$

In this section we give an application of the generalized beta function (2.1) by establishing certain connections in terms of other special functions and polynomials. The results obtained here are interesting and can further be applied to other extensions of beta and other functions.

The multi-index Mittag-Leffler function in (1.10) can be representd directly in the form of Wright generalized hypergeometric function ${}_p\psi_q$ and a generalized form of Fox -H function as follows:

$$E_{\left(\frac{1}{\lambda_i}\right),(\mu_i)}(x) = {}_1\psi_m\left[\begin{matrix}(1,1), \\ \left(\mu_i,\frac{1}{\lambda_i}\right)_1^m\end{matrix}\ x\right] = H_{1,m+1}^{1,1}\left[\begin{matrix}(0,1), & \\ (0,1), & \left(1-\mu_i,\frac{1}{\lambda_i}\right)_1^m\end{matrix}\ -x\right]. \tag{5.1}$$

Using the above representation in (2.1), we write

$$\begin{aligned} B_{p,q}^{\lambda_i,\mu_i;\sigma,\tau}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\, & H_{1,m+1}^{1,1}\left[\begin{matrix}(0,1), & \\ (0,1), & \left(1-\mu_i,\frac{1}{\lambda_i}\right)_1^m\end{matrix}\ \frac{p}{t^\sigma}\right] \\ & \times H_{1,m+1}^{1,1}\left[\begin{matrix}(0,1), & \\ (0,1), & \left(1-\mu_i,\frac{1}{\lambda_i}\right)_1^m\end{matrix}\ \frac{q}{(1-t)^\tau}\right], \end{aligned} \tag{5.2}$$

and

$$\begin{aligned} B_{p,q}^{\lambda_i,\mu_i;\sigma,\tau}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\, & {}_1\psi_m\left[\begin{matrix}(1,1), \\ \left(\mu_i,\frac{1}{\lambda_i}\right)_1^m\end{matrix}\ -\frac{p}{t^\sigma}\right] \\ & \times {}_1\psi_m\left[\begin{matrix}(1,1), \\ \left(\mu_i,\frac{1}{\lambda_i}\right)_1^m\end{matrix}\ -\frac{q}{(1-t)^\tau}\right]. \end{aligned} \tag{5.3}$$

Similarly for $m=2$, the Mittag-Leffler function in (1.10) directly reduces to Dzrbashijan's M-L type function (see [7]) as

$$E_{\left(\frac{1}{\lambda_1},\frac{1}{\lambda_1}\right),(\mu_1,\mu_2)}(x) = \Phi_{\lambda_1,\lambda_2}(x;\mu_1,\mu_2). \tag{5.4}$$

Using above relation in (1.10), we write

$$B_{p,q}^{\lambda_i,\mu_i;\sigma,\tau}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,\Phi_{\lambda_1,\lambda_2}\left(-\frac{p}{t^\sigma};\mu_1,\mu_2\right)\Phi_{\lambda_1,\lambda_2}\left(-\frac{q}{(1-t)^\tau};\mu_1,\mu_2\right). \tag{5.5}$$

Again as a special case of (1.10), in terms of Wright function [1], we can write

$$E_{(\alpha,1),(\beta,1)}(x) = W_{\alpha,\beta}(x) = \sum_{k=0}^{\infty}\frac{1}{\Gamma(\alpha k+\beta)}\frac{x^k}{k!}, \tag{5.6}$$

and consequently get the relation

$$B_{p,q}^{\alpha,\beta;\sigma,\tau}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,W_{\alpha,\beta}\left(-\frac{p}{t^\sigma}\right)W_{\alpha,\beta}\left(-\frac{q}{(1-t)^\tau}\right), \tag{5.7}$$

In other form and denotation, for $m = 2$ in (1.10), in terms of Bessel-Maitland function, we write

$$E_{(\frac{1}{\mu},1),(\nu+1,1)}(-x) = J_\nu^\mu(x) = \sum_{k=0}^{\infty} \frac{(-x)^k}{k!\,\Gamma(\nu + k\mu + 1)}, \tag{5.8}$$

which gives us the representation

$$B_{p,q}^{\mu,\nu;\sigma,\tau}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} J_\nu^\mu\left(-\frac{p}{t^\sigma}\right) J_\nu^\mu\left(-\frac{q}{(1-t)^\tau}\right). \tag{5.9}$$

Now, by considering the generalized Mittag-Leffler function in (1.11), we introduce a particular case of our extended beta function in (2.1). We define

$$B_{p,q}^{\lambda_i;\sigma,\tau}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} E_{(\lambda_i)}\left(-\frac{p}{t^\sigma}\right) E_{(\lambda_i)}\left(-\frac{q}{(1-t)^\tau}\right) dt, \tag{5.10}$$

$$(\Re(x) > 0,\ \Re(y) > 0,\ \Re(p) \geq 0,\ \Re(q) \geq 0,\ \lambda_i, k > 0).$$

The above extended beta function is a typical representative of the following special functions:

We obtain the following relation between $B_{p,q}^{\lambda_i;\sigma,\tau}(x,y)$ and Fox H -function:

$$\begin{aligned} B_{p,q}^{\lambda_i,\mu_i;\sigma,\tau}(x,y) = \int_0^1 & t^{x-1}(1-t)^{y-1} H_{0,2}^{1,0}\left[\frac{p}{t^\sigma} \mid \quad (0;1);\ (0,1),(0,\lambda_i)\right] \\ & \times H_{0,2}^{1,0}\left[\frac{q}{(1-t)^\tau} \mid \quad (0;1);\ (0,1),(0,\lambda_i)\right] dt. \end{aligned} \tag{5.11}$$

By using the relation (see [8]) $J_{0,1}^{\lambda_i,1}(x) = E_{\lambda_i}(x)$ and in view of (2.1), we can write

$$B_{p,q}^{\lambda_i;\sigma,\tau}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} J_{0,1}^{\lambda_i,1}(u)\, J_{0,1}^{\lambda_i,1}(v)\, dt, \tag{5.12}$$

where, $u = -\frac{p}{t^\sigma}$ and $v = -\frac{q}{(1-t)^\tau}$.

The Mittag-Leffler function is connected to the generalized hypergeometric function (see [15]) by the relation

$$E_{\lambda_i,\beta}^{\gamma,q}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}}{\Gamma(\lambda_i n + \beta)} \frac{z^n}{n!} = \frac{1}{\Gamma(\beta)}\, {}_qF_{\lambda_i}\left[\Delta(q;\gamma);\ \Delta(\lambda_i,\beta);\quad \frac{q^q z}{\lambda_i^{\lambda_i}}\right], \tag{5.13}$$

where, $\Delta(\lambda_i,\beta)$ is a q-tuple $\frac{\gamma}{q}, \frac{\gamma+1}{q}, \ldots, \frac{\gamma+q-1}{q}$.
In particular, we have

$$E_{\lambda_i,1}^{1,1}(z) = E_{(\lambda_i)}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\lambda_i n + 1)} = {}_1F_{\lambda_i}\left[\Delta(1;1);\ \Delta(\lambda_i,1);\quad \frac{z}{\lambda_i^{\lambda_i}}\right]. \tag{5.14}$$

Now using (5.14) in (2.1), we have

$$B_{p,q}^{\lambda_i;\sigma,\tau}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}{}_1F_{\lambda_i}\left[\varDelta(1;1);\ \varDelta(\lambda_i,1);\quad \frac{1}{\lambda_i^{\lambda_i}}\left(-\frac{p}{t^{\sigma}}\right)\right] \times {}_1F_{\lambda_i}\left[\varDelta(1;1);\ \varDelta(\lambda_i,1);\quad \frac{1}{\lambda_i^{\lambda_i}}\left(-\frac{q}{(1-t)^{\tau}}\right)\right] dt, \tag{5.15}$$

from which we can write

$$B_{p,q}^{\lambda_i;\sigma,\tau}(x,y) = \int_0^1 \frac{u^{x-1}}{(1+u)^{x+y}}{}_1F_{\lambda_i}\left[\varDelta(1;1);\ \varDelta(\lambda_i,1);\quad \frac{1}{\lambda_i^{\lambda_i}}\left(-\frac{p(1+u)^{\sigma}}{u^{\sigma}}\right)\right] \times {}_1F_{\lambda_i}\left[\varDelta(1;1);\ \varDelta(\lambda_i,1);\quad \frac{1}{\lambda_i^{\lambda_i}}\left(-q(1+u)^{\tau}\right)\right] du. \tag{5.16}$$

6. Extended Hypergeometric and Confluent Hypergeometric functions and their properties

In this section, we present extended forms of hypergeometric and confluent hypergeometric functions in terms of our beta function $B_{p,q}^{\lambda_i,\mu_i;\sigma,\tau}(x,y)$.

$$F_{p,q}^{\lambda_i,\mu_i;\sigma,\tau}(x,y;\omega;z) = \sum_{n=0}^{\infty}(x)_n\,\frac{B_{p,q}^{\lambda_i,\mu_i;\sigma,\tau}(y+n,\omega-y)}{B(y;\omega-y)}\,\frac{z^n}{n!}, \tag{6.1}$$

$$(p \geq 0,\ q \geq 0,\ |z| < 1,\ \lambda_i,\mu_i,\sigma,\tau > 0,\ \Re(\omega) > \Re(y) > 0)$$

and

$$\Phi_{p,q}^{\lambda_i,\mu_i;\sigma,\tau}(y;\omega;z) = \sum_{n=0}^{\infty}\frac{B_{p,q}^{\lambda_i,\mu_i;\sigma,\tau}(y+n,\omega-y)}{B(y;\omega-y)}\,\frac{z^n}{n!}. \tag{6.2}$$

$$(p > 0,\ q > 0,\ \lambda_i,\mu_i > 0,\ \Re(\omega) > \Re(y) > 0)$$

We remark that for $p = q$ and $\forall\lambda_i = \mu_i = 1, \sigma = \tau = 1$, $F_{p,q}^{\lambda_i,\mu_i;\sigma,\tau}(x,y;\omega;z)$ and $\Phi_{p,q}^{\lambda_i,\mu_i;\sigma,\tau}(x,y;\omega;z)$ reduce to (1.6) and (1.7) respectively.

Theorem 7. *The following integral representations for the extended hypergeometric function $F_{p,q}^{\lambda_i,\mu_i;\sigma,\tau}(x,y;\omega;z)$ and confluent hypergeometric function $\Phi_{p,q}^{\lambda_i,\mu_i;\sigma,\tau}(x,y;\omega;z)$ holds true:*

$$F_{p,q}^{\lambda_i,\mu_i;\sigma,\tau}(x,y;\omega;z) = \frac{1}{B(y,\omega-y)} \times \int_0^1 t^{y-1}(1-t)^{\omega-y-1}E_{\left(\frac{1}{\lambda_i}\right),(\mu_i)}\left(-\frac{p}{t^{\sigma}}\right)E_{\left(\frac{1}{\lambda_i}\right),(\mu_i)}\left(-\frac{q}{(1-t)^{\tau}}\right)\sum_{n=0}^{\infty}(x)_n\frac{(zt)^n}{n!}\,dt, \tag{6.3}$$

$(p > 0,\ q > 0;\ \lambda_i,\mu_i > 0;\ p = 0,\ q = 0$ *and* $|z| < 1;\ \Re(\omega) > \Re(y) > 0)$.

$$F_{p,q}^{\lambda_i,\mu_i;\sigma,\tau}(x,y;\omega;z) = \frac{1}{B(y,\omega-y)} \times \int_0^1 t^{y-1}(1-t)^{\omega-y-1}(1-zt)^{-x}\, E_{\left(\frac{1}{\lambda_i}\right),(\mu_i)}\left(-\frac{p}{t^\sigma}\right) E_{\left(\frac{1}{\lambda_i}\right),(\mu_i)}\left(-\frac{q}{(1-t)^\tau}\right) dt, \tag{6.4}$$

$(p > 0,\ q > 0;\ \lambda_i, \mu_i > 0;\ p = 0,\ q = 0$ *and* $|\arg(1-z)| < \pi;\ \Re(\omega) > \Re(y) > 0)$.

$$F_{p,q}^{\lambda_i\mu_i;\sigma,\tau}(x,y;\omega;z) = \frac{1}{B(y,\omega-y)} \times \int_0^\infty u^{y-1}(1+u)^{x-\omega}[u(1-z)]^{-x}\, E_{\left(\frac{1}{\lambda_i}\right),(\mu_i)}\left(-p\left(\frac{1+u}{u}\right)^\sigma\right) E_{\left(\frac{1}{\lambda_i}\right),(\mu_i)}\left(-q(1+u)^\tau\right) du, \tag{6.5}$$

$(p > 0,\ q > 0;\ \lambda_i, \mu_i > 0;\ p = 0,\ q = 0$ *and* $|\arg(1-z)| < \pi;\ \Re(\omega) > \Re(y) > 0)$.

$$F_{p,q}^{\lambda_i,\mu_i;\sigma,\tau}(x,y;\omega;z) = \frac{2}{B(y,\omega-y)} \times \int_0^{\frac{\pi}{2}} \frac{\sin^{2y-1} v\, \cos^{2\omega-2y-1} v}{(1-z\sin^2 v)^x}\, E_{\left(\frac{1}{\lambda_i}\right),(\mu_i)}\left(-p\csc^{2\sigma} v\right) E_{\left(\frac{1}{\lambda_i}\right),(\mu_i)}\left(-q\sec^{2\tau} v\right) dv, \tag{6.6}$$

$(p > 0,\ q > 0;\ \lambda_i, \mu_i > 0;\ p = 0,\ q = 0$ *and* $|\arg(1-z)| < \pi;\ \Re(\omega) > \Re(y) > 0)$.

$$\Phi_{p,q}^{\lambda_i,\mu_i;\sigma,\tau}(y;\omega;z) = \frac{\exp(zt)}{B(y,\omega-y)} \times \int_0^1 t^{y-1}(1-t)^{\omega-y-1}\, E_{\left(\frac{1}{\lambda_i}\right),(\mu_i)}\left(-\frac{p}{t^\sigma}\right) E_{\left(\frac{1}{\lambda_i}\right),(\mu_i)}\left(-\frac{q}{(1-t)^\tau}\right) dt, \tag{6.7}$$

$(p > 0,\ q > 0;\ \lambda_i, \mu_i > 0;\ \Re(\omega) > \Re(y) > 0)$.

$$\Phi_{p,q}^{\lambda_i,\mu_i;\sigma,\tau}(y;\omega;z) = \frac{\exp(z)}{B(y,\omega-y)} \times \int_0^1 t^{y-1}(1-t)^{\omega-y-1} \exp(-zt)\, E_{\left(\frac{1}{\lambda_i}\right),(\mu_i)}\left(-\frac{p}{t^\sigma}\right) E_{\left(\frac{1}{\lambda_i}\right),(\mu_i)}\left(-\frac{q}{(1-t)^\tau}\right) dt. \tag{6.8}$$

$(p > 0,\ q > 0;\ \lambda_i, \mu_i > 0;\ \Re(\omega) > \Re(y) > 0)$.

Proof. We can easily obtain (6.3) by using the definition (2.1) in (6.1). The integral (6.4) can be obtained by using the binomial expansion

$$(1-zt)^{-x} = \sum_{n=0}^{\infty} (x)_n \frac{(zt)^n}{n!}$$

in (6.3). By choosing $t=\frac{u}{1+u}$, $t=\sin^2 v$ in (6.4), we obtain (6.5) and (6.6) respectively. By using a similar approach, we can easily establish (6.7) and (6.8). □

Remark 6.1. *The case* $\sigma=\tau=\lambda_i=\mu_i=1$ *and* $\lambda_i=\mu_i=1$, $p=q$, $\sigma=\tau$ *in equations* (6.3)-(6.8) *leads to the corresponding results in* [6] *and* [11], *respectively. For* $p=0=q$ *and* $\sigma=\tau=\lambda_i=\mu_i=1$, *we get basic hypergeometric and confluent hypergeometric function* [1].

Theorem 8. *Let* $\Re(p)\geq 0$, $\Re(q)\geq 0$ *and* $\lambda_i\in\mathbb{C}$. *The extended hypergeometric function* $F_{p,q}^{\lambda_i,\mu_i;\sigma,\tau}$ *possesses the following generating function:*

$$\begin{aligned}&\sum_{n=0}^{\infty}\binom{\zeta+n-1}{n}F_{p,q}^{\lambda_i,\mu_i;\sigma,\tau}(\zeta+n,y;\omega;z)\,t^n\\&=(1-t)^{-\lambda_i}F_{p,q}^{\lambda_i,\mu_i;\sigma,\tau}\left(\lambda_i,y;\omega;\frac{z}{1-t}\right)\quad(t\leq 1).\end{aligned}\tag{6.9}$$

Proof. We recall the generalized binomial coefficient

$$\binom{\zeta}{\vartheta}:=\frac{\Gamma(\zeta+1)}{\Gamma(\vartheta+1)\Gamma(\zeta-\vartheta+1)}=:\binom{\zeta}{\zeta-\vartheta}\quad(\zeta,\vartheta\in\mathbb{C}),\tag{6.10}$$

such that for $\vartheta=n$ $(n\in\mathbb{N}_0)$, we get

$$\binom{\zeta}{n}=\frac{\zeta(\zeta-1)\cdots(\zeta-n+1)}{n!}=\frac{(-1)^n(-\zeta)_n}{n!}\quad(n\in\mathbb{N}_0).\tag{6.11}$$

Now, let L be the left hand side of assertion (6.9). Using (6.1) into L, we get

$$\begin{aligned}L=&\sum_{n=0}^{\infty}\binom{\zeta+n-1}{n}\left(\sum_{k=0}^{\infty}(\zeta+n)_k\,\frac{B_{p,q}^{\lambda_i,\mu_i;\sigma,\tau}(y+k,\omega-y)}{B(y,\omega-y)}\,\frac{z^k}{k!}\right)t^n\\=&\sum_{k=0}^{\infty}(\zeta)_k\,\frac{B_{p,q}^{\lambda_i,\mu_i;\sigma,\tau}(y+k,\omega-y)}{B(y,\omega-y)}\left[\sum_{n=0}^{\infty}\binom{\zeta+n+k-1}{n}t^n\right]\frac{z^k}{k!}.\end{aligned}\tag{6.12}$$

Finally, applying the generalized binomial expansion

$$\sum_{n=0}^{\infty}\binom{\zeta+n-1}{n}t^n=(1-t)^{-\zeta}\quad(|t|<1;\zeta\in\mathbb{C}),$$

on the inner summation in (6.12), we get the expected result (6.9). □

7. Differentiation formulas for $F_{p,q}^{\lambda_i,\mu_i;\sigma,\tau}(x,y;\omega;z)$ and $\Phi_{p,q}^{\lambda_i,\mu_i;\sigma,\tau}(x,y;\omega;z)$

Lemma 7.1. *Consider the following relation for beta function:*

$$B(y,\omega-x)=\frac{\omega}{y}\,B(y+1,\omega-y).\tag{7.1}$$

Theorem 9. *For $n \in \mathbb{N}_0$, the following differentiation formulas holds true:*

$$\frac{d}{dz}\left\{F_{p,q}^{\lambda_i,\mu_i;\sigma,\tau}(x,y;\omega;z)\right\} = \frac{xy}{\omega}\, F_{p,q}^{\lambda_i,\mu_i;\sigma,\tau}(x+1,y+1;\omega+1;\ z). \tag{7.2}$$

$$\frac{d^n}{dz^n}\left\{F_{p,q}^{\lambda_i,\mu_i;\sigma,\tau}(x,y;\omega;z)\right\} = \frac{(x)_n(y)_n}{(\omega)_n}\, F_{p,q}^{\lambda_i,\mu_i;\sigma,\tau}(x+n,y+n;\omega+n;\ z). \tag{7.3}$$

$$\frac{d^n}{dz^n}\left\{\Phi_{p,q}^{\lambda_i,\mu_i;\sigma,\tau}(y;\omega;z)\right\} = \frac{(y)_n}{(\omega)_n}\, \Phi_{p,q}^{\lambda_i,\mu_i;\sigma,\tau}(x+n,y+n;\omega+n;\ z). \tag{7.4}$$

Proof. By differentiating (6.1) with repect to z, we get

$$\frac{d}{dz}\, F_{p,q}^{\lambda_i,\mu_i;\sigma,\tau}(x,y;\omega;z) = \sum_{n=1}^{\infty} \frac{B_{p,q}^{\lambda_i,\mu_i;\sigma,\tau}(y+n,\omega-y)}{B(y,\omega-y)}\,(x)_n\,\frac{z^{n-1}}{(n-1)!}.$$

On replacing n by $n+1$ and using (6.4) and (7.1), we easily get (7.2). A recursive process of this establishes (7.3). In a similar way, we can obtain (7.4). □

Transformation formulas for $F_{p,q}^{\lambda_i,\mu_i;\sigma,\tau}(x,y;\omega;\ z)$ and $\Phi_{p,q}^{\lambda_i,\mu_i;\sigma,\tau}(x,y;\omega;\ z)$

We have the following transformation formulas for the extended hypergeometric and confluent hypergeometric functions:

$$F_{p,q}^{\lambda_i,\mu_i;\sigma,\tau}(x,y;\omega;\ z) = (1-z)^{-\alpha}\, F_{p,q}^{\lambda_i,\mu_i;\sigma,\tau}\left(x,\omega-y;\omega;\ -\frac{z}{1-z}\right). \tag{7.5}$$

$(p>0, q>0;\ p=0, q=0$ and $\arg(1-z)<\pi;\ \Re(\omega)>\Re(y)>0)$.

$$F_{p,q}^{\lambda_i,\mu_i;\sigma,\tau}\left(x,y;\omega;\ 1-\frac{1}{z}\right) = z^{\alpha}\, F_{p,q}^{\lambda_i,\mu_i;\sigma,\tau}(x,\omega-y;\omega;\ 1-z). \tag{7.6}$$

$(p>0, q>0;\ p=0, q=0$ and $\arg(1-z)<\pi;\ \Re(\omega)>\Re(y)>0)$.

$$F_{p,q}^{\lambda_i,\mu_i;\sigma,\tau}\left(x,y;\omega;\ \frac{z}{1+z}\right) = (1+z)^{\alpha}\, F_{p,q}^{\lambda_i,\mu_i;\sigma,\tau}(x,\omega-y;\omega;\ -z). \tag{7.7}$$

$(p>0, q>0;\ p=0, q=0$ and $\arg(1-z)<\pi;\ \Re(\omega)>\Re(y)>0)$.

$$\Phi_{p,q}^{\lambda_i,\mu_i;\sigma,\tau}(y,\omega;\ z) = e^{z}\, \Phi_{p,q}^{\lambda_i,\mu_i;\sigma,\tau}(\omega-y;\omega;\ -z). \tag{7.8}$$

Proof. Using the expression

$$[1-z(1-t)]^{-x} = (1-z)^{-x}\left(1+\frac{z}{1-z}t\right)^{-x},$$

and replacing t by $1-t$ in (6.4), we have

$$F_{p,q}^{\lambda_i,\mu_i;\sigma,\tau}(x,y;\omega;z)=\frac{(1-z)^{-x}}{B(y,\omega-y)}$$
$$\times\int_0^1 t^{y-1}(1-t)^{\omega-y-1}\left(1+\frac{z}{1-z}t\right)^{-x}E_{\left(\frac{1}{\lambda_i}\right),(\mu_i)}\left(-\frac{p}{t^\sigma}\right)E_{\left(\frac{1}{\lambda_i}\right),(\mu_i)}\left(-\frac{q}{(1-t)^\tau}\right)dt \tag{7.9}$$

$$(p>0,\ q>0;\ \lambda_i,\mu_i>0;\ p=0,\ q=0\ \textit{and}\ |\arg(1-z)|<\pi;\ \Re(\omega)>\Re(y)>0),$$

which easily proves (7.5). Replacing z by $1-\frac{1}{z}$ and $\frac{z}{1+z}$ in (7.5) yields (7.6) and (7.7) respectively. The formula (7.8) can be obtained by following (6.7) and (6.8). □

Theorem 10. *The extended hypergeometric function* $F_{p,q}^{\lambda_i,\mu_i;\sigma,\tau}(x,y;\omega;z)$ *has the following Mellin transformation formula:*

$$M\left\{F_{p,q}^{\lambda_i,\mu_i;\sigma,\tau}(x,y;\omega;z);p\to r,q\to s\right\}=\frac{\pi^2}{sin(\pi r)\,sin(\pi s)\Gamma(\mu_i-r/\lambda_i)\Gamma(\mu_i-s/\lambda_i)}$$
$$\times\frac{B(y+r,\omega+s-y)}{B(y,\omega-y)}F(x,y+r,\omega+r+s;z) \tag{7.10}$$
$$(\Re(y+r)>0,\ \Re(\omega+s)>0).$$

Proof. We begin by providing Euler's reflection formula, which we use later in theorem

$$\Gamma(x)\,\Gamma(1-x)=\frac{\pi}{\sin(\pi x)}. \tag{7.11}$$

Applying the usual Mellin transform on (6.1), we get

$$M\left\{F_{p,q}^{\lambda_i,\mu_i;\sigma,\tau}(x,y;\omega;z);p\to r,q\to s\right\}=\frac{1}{B(y,\omega-y)}\int_0^\infty\int_0^\infty p^{r-1}q^{s-1}$$
$$\times\left\{\int_0^1 t^{y-1}(1-t)^{\omega-y-1}(1-zt)^{-x}E_{\left(\frac{1}{\lambda_i}\right),(\mu_i)}\left(-\frac{p}{t^\sigma}\right)E_{\left(\frac{1}{\lambda_i}\right),(\mu_i)}\left(-\frac{q}{(1-t)^\tau}\right)dt\right\}dp\,dq.$$

Interchanging the order of integrations, we have

$$M\left\{F_{p,q}^{\lambda_i,\mu_i;\sigma,\tau}(x,y;\omega;z);p\to r,q\to s\right\}=\frac{1}{B(y,\omega-y)}\int_0^1 t^{y-1}(1-t)^{\omega-y-1}(1-zt)^{-x}$$
$$\times\left\{\int_0^\infty p^{r-1}E_{\left(\frac{1}{\lambda_i}\right),(\mu_i)}\left(-\frac{p}{t^\sigma}\right)dp.\int_0^\infty q^{s-1}E_{\left(\frac{1}{\lambda_i}\right),(\mu_i)}\left(-\frac{q}{(1-t)^\tau}\right)dq\right\}dt.$$

Now substituting $\frac{p}{t^\sigma}=u$ and $\frac{q}{(1-t)^\tau}=v$ above, we obtain

$$M\left\{F_{p,q}^{\lambda_i,\mu_i;\sigma,\tau}(x,y;\omega;z);p\to r,q\to s\right\}=\frac{1}{B(y,\omega-y)}\int_0^1 t^{y+r\sigma-1}(1-t)^{\omega+\tau s-y-1}$$
$$\times(1-zt)^{-x}\left\{\int_0^\infty u^{r-1}E_{\left(\frac{1}{\lambda_i}\right),(\mu_i)}(-u)\,du.\int_0^\infty v^{s-1}E_{\left(\frac{1}{\lambda_i}\right),(\mu_i)}(-v)\,dv\right\}dt,$$

which, on using (1.13) gives

$$M\left\{F_{p,q}^{\lambda_i,\mu_i;\sigma,\tau}\left(x,y;\omega;z\right);p\to r,q\to s\right\}$$
$$=\frac{\Gamma(r)\Gamma(1-r)}{\Gamma(\mu_i-r/\lambda_i)}\frac{\Gamma(s)\Gamma(1-s)}{\Gamma(\mu_i-s/\lambda_i)}\ \frac{B(y+\sigma r,\omega+\tau s-y)}{B(y,\omega-y)}\,F(x,y+\sigma r,\omega+\sigma r+\tau s;z).$$

Taking into account the formula (7.11) leads us to the result (7.10). □

Theorem 11. *The following Mellin transformation formula holds:*

$$M\left\{\Phi_{p,q}^{\lambda_i,\mu_i;\sigma,\tau}\left(y;\omega;z\right);p\to r,q\to s\right\}=\frac{\pi^2}{sin(\pi r)\,sin(\pi s)\Gamma(\mu_i-r/\lambda_i)\Gamma(\mu_i-s/\lambda_i)}$$
$$\times\frac{B(y+\sigma r,\omega+\tau s-y)}{B(y,\omega-y)}\,\Phi(y+\sigma r,\omega+\sigma r+\tau s;z). \tag{7.12}$$
$$(\Re(y+r)>0,\ \Re(\omega+s)>0).$$

8. **Conclusion**

In our present investigation, we first introduced an extension of beta function by means of a 2-parameter multi-index Mittag-Leffler function defined by (1.10). After an exhaustive study of this extended beta function defined in (2.1), we introduced an extension of the hypergeometric and confluent hypergeometric function. Eventually, we presented a systematic study of the various fundamental properties of these functions introduced here.

We remark that from the special cases of our main definitions, several definitions of beta, hypergeometric and confluent hypergeometric function and their corresponding properties directly follows.

References

[1] G.E. Andrews, R. Askey, R. Roy, *Special functions*, Cambridge University Press, Cambridge, 1999.

[2] M.A. Chaudhry, A. Qadir, M. Rafique and S.M. Zubair, Extension of Euler's beta function, *J. Comput. Appl. Math.*, **78** (1997), 19–32.

[3] M.A. Chaudhry, A. Qadir, H.M. Srivastava and R.B. Paris, Extended Hypergeometric and Confluent Hypergeometric functions, *Appl. Math. Comput.*, **159** (2004), 589–602.

[4] M.A. Chaudhry, N.M. Temme, E.J.M. Veling, Asymptotic and closed-form of a generalized incomplete gamma function, *J. Comput. Appl. Math.*, **67** (1996), 371–379.

[5] M.A. Chaudhry, S.M. Zubair, Generalized incomplete gamma functions with applications, *J. Comput. Appl. Math.*, **55** (1994), 99–124.

[6] J. Choi, A.K Rathie, R.K. Parmar, Extension of extended beta, hypergeometric and confluent hypergeometric functions, *Honam Mathematical Journal*, **36**(2) 2014, 357–385.

[7] M. Dzrbashjan, On the integral transformations generated by the generalized Mittag-Leffler function, *Izv. Akad. Nauk Armen. SSR* **13** (1960), 21–63. (in Russian).

[8] N.U. Khan, T. Kashmin, Some integrals for the generalized Bessel Maitland functions, *Electronic Journal of Mathematical Analysis and Applications*, **4**(2) (2016), 139–149.

[9] N.U. Khan, T. Usman and Mohd Aman, Generalized Wright function and its properties using extended beta function, *Tamkang journal of Mathematics*, **51**(4) (2020) 349–363.

[10] V. Kiryakova, The multi-index Mittag-Leffler functions as an important class of special functions of fractional calculus, *Computers and Mathematics with Applications*, **59** (2010), 1885–1895.

[11] D.M. Lee, A.K. Rathie, R.K. Parmar and Y.S. Kim, Generalization of extended beta function, hypergeometric and confluent hypergeometric functions, *Honam Mathematical J.*, **33**(2) (2011), 187–206.

[12] A.M. Mathai, H.J. Haubold, *Special functions for applied scientists*, New York: Springer, 2008.

[13] E. Özerjin, M.A. Özarslan, A. Altin, Extension of gamma, beta and hypergeometric functions, *J. Comput. Appl. Math.*, **235** 2011, 4601–4610.

[14] M. Shadab, S. Jabee, J. Choi, An extended beta function and its applications, *Far East Journal of Mathematical Sciences*, **103**(1) 2018, 235–251.

[15] A.K. Shukla, J.C. Prajapati, On a generalization of Mittag-Leffler function and its properties, *J. Math. Anal. Appl.*, **336**, (2007), 797–811.

[16] H.M. Srivastava, P.W. Karlsson, *Multiple Gaussian Hypergeometric Series*, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1985.

Nabiullah Khan: Department of Applied Mathematics, Faculty of Engineering and Technology, Aligarh Muslim University, Aligarh 202002, India
Email address: nukhanmath@gmail.com

Mohd Aman: Department of Applied Mathematics, Faculty of Engineering and Technology, Aligarh Muslim University, Aligarh 202002, India
Email address: mohdaman.maths@gmail.com

Talha Usman: Department of Mathematics, School of Basic and Applied Sciences, Lingaya's Vidyapeeth, Faridabad-121002, Haryana, India

Department of General Requirements, University of Technology and Applied Sciences-Sur, Sultanate of Oman
Email address: talhausman.maths@gmail.com